Geodesics of Kerr Space-time: Equatorial Geodesics of Kerr Black Hole

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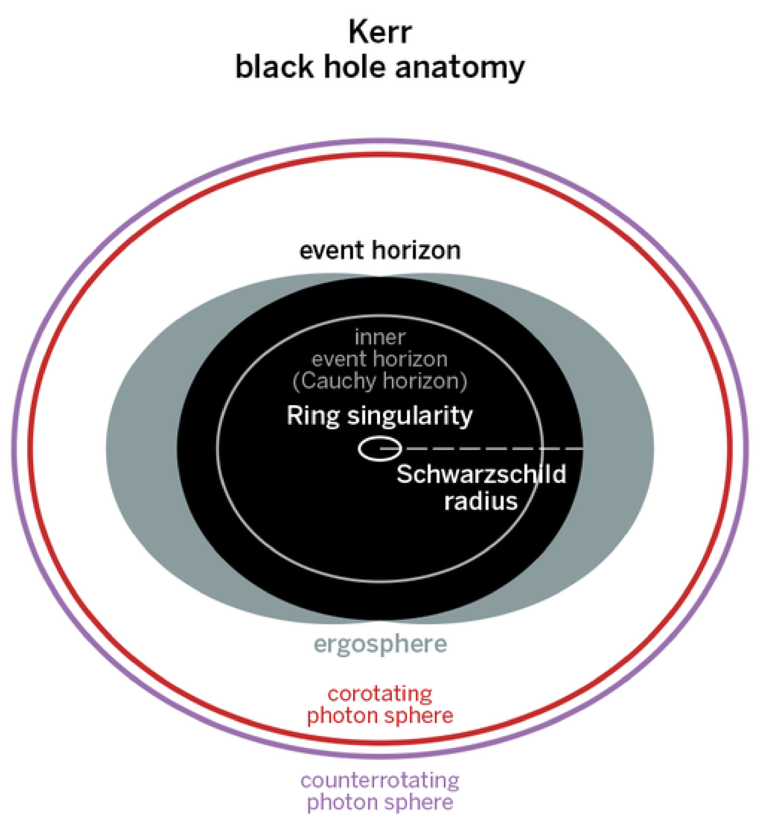
1. Abstract

This paper seeks to study the geodesics motion outside a Kerr black hole (i.e. restricted to the region outside the outer horizon). With a basic understanding of the concepts in differential geometry, namely smooth manifolds, tensors, semi-Riemannian geometry, and the basics of geodesics and Kerr metrics, this paper then discussed equatorial geodesics, the geodesics with $\theta \equiv \frac{\pi}{2}$. With detailed outlining about the properties of equatorial geodesics, the paper then discussed about more general case, where *Hamilton-Jacobi approach* is applied.

2. Background

In general relativity, a geodesic is a generalization of the notion of a "straight line" to curved spacetime. In particular, the world line(frame) of a particle free from all external, non-gravitational force is a particular type of geodesic.

To mathematically analyze its properties, we employed a system of metric, the Kerr metric. According to the Kerr metric, rotating black-holes should exhibit frame-dragging, a distinctive prediction of general relativity. This effect predicts that objects coming close to a rotating mass will be entrained to participate in its rotation, because of the swirling curvature of space-time itself associated with rotating bodies. At close enough distances, all objects - even light - must rotate with the black-hole.



3. Geodesics

First we generalize the Euclidean notion of straight line. A *geodesic* in a semi-Riemannian manifod M is a curve $\gamma : I \to M$ whose vector field γ' is parallel. Equivalently, geodesics are the curves of acceleration zero $\gamma'' = 0$.

Definition 1. Let $x^1, ..., x^n$ be a coordinate system on $u \subset M$. A curve γ in U is a geodesic of M if and only if its coordinate functions $x^k \circ \gamma$ satisfy

$$\frac{d^2(x^k \circ \gamma)}{dt^2} + \sum i, j\Gamma^k_{ij}(\gamma)\frac{d(x^i \circ \gamma)}{dt}\frac{d(x^j \circ \gamma)}{dt} = 0$$

for $1 \leq k \leq n$.

4. Equatorial Geodesics

Equatorial geodesics are defined as geodesics with:

$$\theta \equiv \frac{\pi}{2}.$$

We first find that such geodesics exist, i.e. that equatorial geodesics are solutions of the geodesics equation, or equivalently, of the Euler-Lagrange equations, which is given by

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{\partial L}{\partial x^\alpha}.$$

If, at $\lambda = 0$, the particles moves in the equatorial plane, $\theta(\lambda = 0) = \frac{\pi}{2}$ and $\dot{\theta}(\lambda = 0) = 0$; then we have a well-posed Cauchy problem of the form:

$$\ddot{\theta} = \cdots; \dot{\theta}(\lambda = 0) = 0; \theta(\lambda = 0) = \frac{\pi}{2}$$

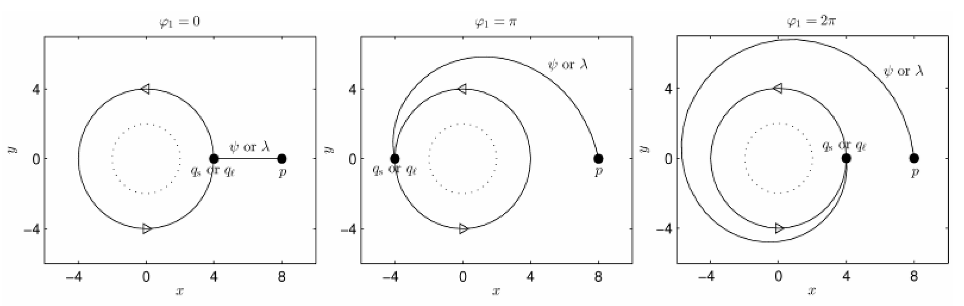
which admits one and only one solution; since $\theta \equiv \frac{\pi}{2}$ is a solution, it is the solution. Thus a geodesic which starts in the equatorial plane, remains in the equatorial plane.

It can be shown that this also happens in the Schwarzschild metric, and it is possible to generalize the result to any orbit. In the case of Kerr metric, however, the generalization to any orbit is not possible. This is because unlike Schwarzschild metric, which is planar, Kerr metric is only axially symmetric. We can, however, conclude that geodesics strating in the equatorial plane are planar.

Under this case, it can be shown that the equation in the energy state,E, can be solved via the equation:

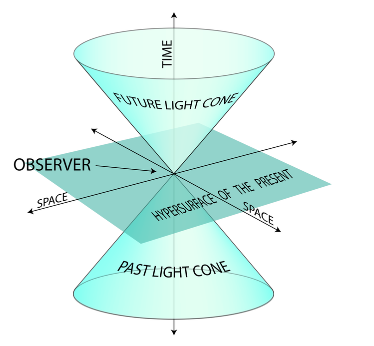
$$CE^2 - 2BLE - AL^2 = 0,$$

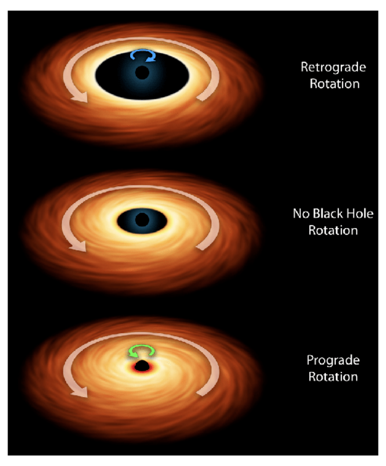
where $A \equiv 1 - \frac{2M}{r}, B \equiv \frac{2Ma}{r}, C \equiv r^2 + a^2 + \frac{2Ma^2}{r}$



figureDiagrams in the xy-plane of different equatorial geodesics joining the test particle and the observer at p The parameter ϕ_1 represents the ϕ-coordinate of q_s

5. Timelike Geodesics





In the case of timelike geodesics, the equation becomes:

$$\dot{r}^2 = \frac{C}{r^2}(E - V_+)(E - V_-) - \frac{\Delta}{r^2},$$

And under $\kappa = -1$ This equation would not allow a simple qualitative study as in the case of null geodesics. Therefore, we would restrict our attention to a very relevant quantity (with astrophysical interest), namely the location of the innermost stable circular orbit (ISCO), which, in the Schwarzschild case, is at $r = 6M$. In Kerr spacetime, the qualitative behavior for r_{ISCO} is simple: there are two solutions

$$r^{\pm}_{ISCO}(a)$$

one corresponding to corotating orbits, one to counterrotating orbits. For $a = 0$, obviously the two solutions coincide to $6M$; by increasing $|a|$, the ISCO moves closer to the black hole for corotating orbits, and far away from the black hole for counterrotating orbits. It can also be verified that a circular timelike geodesic in the equatorial plane satisfies the $3^r d$ Kepler law. We remind that the Lagrangian is

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$$

and the r Euler-Lagrange equation, being $g_{r\mu} = 0$ if $\mu \neq r$, is

$$\frac{d}{d\lambda}(g_{rr}\dot{r}) = \frac{1}{2}g_{\mu\nu,r}\dot{x}^\mu \dot{x}^\nu$$

and note that the solutions

$$\omega_{pm} = \frac{\sqrt{M}}{r^{\frac{3}{2}} \pm a\sqrt{M}}$$

This is the relation between angular velocity and radius of circular orbits, and reduces, in Schwarzschild limit $a = 0$, to

$$\omega_{pm} = \pm\sqrt{\frac{M}{r^3}},$$

which is Kepler's $3^r d$ law.

6. Future works

This research presented a wide range of materials in differential geometry that is essential for the understanding of the works in Kerr Metrics. The materials covered including smooth manifolds, tensors, semi-Riemannian manifolds, parallel translations, geodesics and Kerr metrics.
With an understanding of all the essential materials covered in the Kerr Black metrics, the upcoming work is likely to have a heavier focus on discussing:

1. the key materials central to the bound in non-rotating black hole..
2. an approach moving from the specific case of geodesics (the equatorial geodesics) to a more general case (the geodesics with varying angle).

In light of all the previous research, it is hoped that an better approximation on the current bond could be obtained by the end of this research.

7. References

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